A Polyhedral Study of the Triplet Formulation for Single Row Facility Layout Problem

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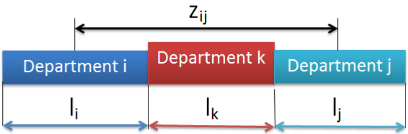
Outline

Single Row Facility Layout Problem (SRFLP)



- l_i - Length of department i
- c_{ij} - Average daily traffic between departments i and j
- z_{ij}^{π} - Distance between centroids of departments i and j in permutation π

Objective of SRFLP:

$$\min_{\pi} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c_{ij} z_{ij}^{\pi}$$

Features of SRFLP:

- SRFLP - generalization of *Minimum Linear Arrangement Problem*, NP-hard.
- Solution techniques: branch and bound, dynamic programming, non-linear programming, semi-definite programming, linear mixed-integer programming.
- MIP formulations of SRFLP:
  - Distance polytope formulation (Amaral and Letchford (2010)): d_{ij}, distance between departments i and j
  - Triplet polytope formulation (Amaral (2009)): binary decision variables

Motivation for study

- Amaral's MIP formulation of SRFLP - triplet formulation (Amaral (2009)).
- Two projections of triplet formulation and its valid inequalities presented in Amaral (2009).
- Linear program solved over these valid inequalities yields optimal solution for several classical instances of size $n = 5$ to $n = 30$.

Research contributions:

- Dimension of the triplet polytope is $n'' = n(n-1)(n-2)/3$.
- Almost all valid inequalities defined for the triplet polytope by Amaral are facet-defining.
- The above results are also true for the other projections of the triplet polytope defined in Amaral (2009).

Polyhedra revisited

- Region enclosed by black lines and axes: $\{x \in \mathcal{R}_+^2 : Ax \leq b\}$
- Region enclosed by red lines: $\{x \in \mathcal{Z}_+^2 : Ax \leq b\}$ - MIP convex hull
- Inequalities describing MIP convex hull - facet-defining

Variables and Parameters

- Set of departments: $N = \{1, ..., n\}$
- Decision variable:

$$\lambda_{ijk} = \begin{cases} 1 \text{ if dept } k \text{ lies between departments } i \text{ and } j,\ i < j \\ 0 \text{ otherwise.} \end{cases}$$

For the above permutation, $\lambda_{ijk} = 0, \lambda_{ikj} = 1, \lambda_{jki} = 0$.

- Decision variable vector: $\lambda = \{\lambda_{ijk} : i, j, k \in N, i < j\}$

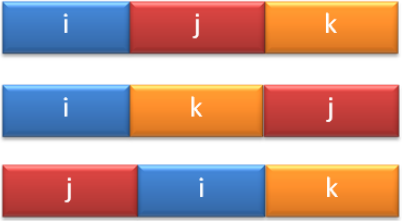
Variables and Parameters

- Number of elements of λ: $n' = n(n-1)(n-2)/2$
- $P^1 = \{\lambda \in \{0,1\}^{n'} \colon \lambda \text{ represents a permutation of } \{1,...,n\}\}$
- *Triplet polytope:* convex hull of P^1.

Objective function of SRFLP: $\min \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c_{ij} \left(\frac{1}{2}(l_i + l_j) + \sum_{k \neq i, k \neq j}^{n} l_k \lambda_{ijk} \right)$

Valid inequalities for SRFLP

$$0 \leq \lambda_{ijk} \leq 1 \qquad i, j, k \in N, i < j$$
$$\lambda_{ijk} + \lambda_{ikj} + \lambda_{jki} = 1 \qquad i, j, k \in N, i < j < k \tag{1}$$



Valid inequalities for SRFLP

$$\lambda_{jkd} + \lambda_{ikd} \geq \lambda_{ijd} \qquad i, j, k, d \in N, i < j < k \tag{2}$$

Here, $\lambda_{ijd} = 1, \lambda_{ikd} = 1, \lambda_{jkd} = 0$.

Similarly,

$$\lambda_{ijd} + \lambda_{jkd} \geq \lambda_{ikd} \qquad i, j, k, d \in N, i < j < k \tag{3}$$

$$\lambda_{ijd} + \lambda_{ikd} \geq \lambda_{jkd} \qquad i, j, k, d \in N, i < j < k \tag{4}$$

$$\lambda_{ijd} + \lambda_{jkd} + \lambda_{ikd} \leq 2 \qquad i, j, k, d \in N, i < j < k \tag{5}$$

T-S inequalities

- Consider $S \subseteq N$.
- Let $T \subseteq N \backslash S$ such that $|T| = |S| - 1$.
- Consider a department $r \in N$, but not in S or T.
- Valid inequality developed using sets S, T and department r:

$$\sum_{p,q \in S : p<q} \lambda_{pqr} + \sum_{p,q \in T : p<q} \lambda_{pqr} \leq \sum_{p \in S, q \in T} \lambda_{pqr}$$

$Conv(P^1)$

Theorem 1

$conv(P^1)$ is of dimension $n'' = n(n-1)(n-2)/3$.

- No. of variables used to describe P^1: $n' = n(n-1)(n-2)/2$.
- No. of linearly independent equalities: $\binom{n}{3}$ (of the form $\lambda_{ijk} + \lambda_{ikj} + \lambda_{jki} - 1 = 0, i < j < k$).
- Consider any other hyperplane of the form

$$\sum_{i,j,k \in N: i<j} a_{ijk}\lambda_{ijk} = b. \tag{6}$$

Lemma 1

Lemma 1. *For given distinct departments $x, y, z \in N$, let $\pi^1, \pi^2, \pi^3, \pi^4$ be four permutations of the departments in N satisfying the following conditions:*

- π^1: $(x, y, \ldots)$
- π^2: $(y, x, \ldots)$
- π^3: $(z, x, y, \ldots)$
- π^4: $(z, y, x, \ldots)$

If the λ vectors corresponding to these permutations lie on the hyperplane

$$\sum_{i,j,k \in N : i<j} a_{ijk}\lambda_{ijk} = b, \tag{7}$$

then $a_{yzx} = a_{xzy}$.

$Conv(P^1)$

- From lemma 1, for any $i, j, k \in N$ we have $a_{ijk} = a_{ikj} = a_{jki}$.
- Using these relationships between coefficients, the hyperplane can be reduced to,

$$\sum_{i,j,k \in N : i<j} a_{ijk}[\lambda_{ijk} + \lambda_{jki} + \lambda_{ikj} - 1] = 0.$$

- $Dim(Conv(P^1)) = n' - \binom{n}{3} = n''$, where $n'' = n(n-1)(n-2)/3$.

Valid inequalities of P^1

Theorem 2

The T-S inequalities are facet-defining for $conv(P^1)$.

T-S inequalities: For any $S \subseteq N$, $T \subseteq N \backslash S$ with $|T| = |S| - 1$ and a department $r \in N \backslash \{S \cup T\}$,

$$\sum_{p,q \in S : p<q} \lambda_{pqr} + \sum_{p,q \in T : p<q} \lambda_{pqr} \leq \sum_{p \in S, q \in T} \lambda_{pqr}$$

- Choosing $S = \{i, j\}, T = \{k\}$ and any other department d we get,

$$\lambda_{ijd} \leq \lambda_{ikd} + \lambda_{jkd}.$$

- We consider the face of the triplet polytope in which the above inequality holds at equality.

Valid inequalities of P^1

Sketch of proof:

- Number of variables used to describe face: n'.
- Number of LI equalities currently known: $\binom{n}{3} + 1$.
- Considering a general hyperplane,

$$\sum_{i,j,k \in N : i<j} a_{ijk} \lambda_{ijk} = b.$$

- By using lemma 1 several times, we develop relationships between coefficients of this hyperplane.
- With these relationships, we prove that the above hyperplane cannot be a LI equality.
- Hence, Dimension of the face in which a T-S inequality holds at equality $= n' - (\binom{n}{3} + 1)) = n'' - 1$.

Projections of P^1

- Amaral defines two other projections of the triplet polytope and its VIs.
- These results can be established for the projections using affine independence.

Conclusions

- Convex hull of triplet polytope and its projections are of dimension $n(n-1)(n-2)/3$.
- Several valid inequalities presented for the triplet polytope by Amaral are facet-defining.
- Theoretical support for computational results in Amaral (2009).

Major References

1. Amaral, A. R. S. 2009. A new lower bound for the single row facility layout problem. *Discrete Applied Mathematics* **157(1)**, 183–190.
2. **Sanjeevi, S.** and **Kianfar, K.** 2010. A polyhedral study of triplet formulation for single row facility layout problem. **Discrete Applied Mathematics** 158(16), 1861–1867.

Questions?