A New Perspective on Generating Facets for Polyhedral Second-Order Conic Sets

Sujeevraja Sanjeevi and Kiavash Kianfar

Industrial and Systems Engineering
Texas A&M University

INFORMS 2012 - Phoenix, AZ

October 15, 2012

Outline

SOCMIP

Second-order Conic Mixed Integer Program:

$$
\begin{aligned}
&\min\ cx + ry \\
&\text{s.t. } \|A_i x + G_i z - b_i\| \leq d_i x + e_i z - h_i, \quad i = 1, \ldots, k, \\
&x \in \mathbb{Z}^n, z \in \mathbb{R}^p
\end{aligned}
$$

- A_i, G_i, and b_i have m_i rows
- d_i, e_i, c and r are vectors and h_i is a scalar
- Rational data

PSOC (Atamtürk and Narayanan, 2010)

Set with conic constraint:

$$X = \left\{(x, z) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p : \|Ax + Gz - b\| \leq dx + ez - h\right\}$$

Reformulation:

$$t_0 \leq dx + ez - h \tag{1}$$

$$t_i \geq |a_i x + g_i z - b| \quad i = 1, ..., m \tag{2}$$

$$t_0 \geq \|t\| \tag{3}$$

Inequality (2): Polyhedral second-order conic (PSOC) constraint

Simple conic MIR inequality (Atamtürk and Narayanan, 2010)

$$S_0 := \left\{(x, w^+, w^-, t) \in \mathbb{Z} \times \mathbb{R}^2_+ \times \mathbb{R} : |x + w^+ - w^- - b| \leq t\right\}.$$

Simple conic MIR inequality (Atamtürk and Narayanan, 2010):

$$(1 - 2f)\,(x - \lfloor \beta \rfloor) + f \leq t + w^+ + w^-$$

where $f = \beta - \lfloor \beta \rfloor$.

- Valid for S_0, cuts off all points in $relax(S_0) \setminus conv(S_0)$.
- Generates nonlinear inequalities for S_0.

Simple conic MIR inequality (Atamtürk and Narayanan, 2010)

$$S_0 := \left\{(x, w^+, w^-, t) \in \mathbb{Z} \times \mathbb{R}^2_+ \times \mathbb{R} : |x + w^+ - w^- - \beta| \leq t\right\}.$$

Simple conic MIR inequality (Atamtürk and Narayanan, 2010):

$$(1-2f)\,(x - \lfloor\beta\rfloor) + f \leq t + w^+ + w^- \tag{4}$$

where $f = \beta - \lfloor\beta\rfloor$.

Sketch of proof of validity:

- Consider $x = \lfloor\beta\rfloor - \alpha$ for $\alpha \geq 0$.
  - Defining inequality becomes $t \geq |w^+ - w^- - f - \alpha|$.
  - (4) becomes $t \geq -w^+ - w^- + f - \alpha(1-2f)$.
  - $|w^+ - w^- - f - \alpha| - (-w^+ - w^- + f - \alpha(1-2f)) \geq 0$.
- Similar proof when $x = \lceil\beta\rceil + \alpha$.

General PSOC set (Atamtürk and Narayanan, 2010)

$$S := \left\{ (x, w^+, w^-, t) \in \mathbb{Z}_+^N \times \mathbb{R}_+^2 \times \mathbb{R} : \left| \sum_{j \in J} a_j x_j + w^+ - w^- - b \right| \leq t \right\}$$

Conic MIR inequality for S:

$$\sum_{j \in J} \phi_f(a_j) x_j - \phi_f(b) \leq t + w^+ + w^-$$

where $\phi_f : \mathbb{R} \to \mathbb{R}$ is the conic MIR function ($f = b - \lfloor b \rfloor$):

$$\phi_f(a) = \begin{cases} (1-2f)n - (a-n), & n \leq a < n+f \\ (1-2f)n + (a-n) - 2f, & n+f \leq a < n+1 \end{cases} \quad n \in \mathbb{Z}$$

Conic MIR function

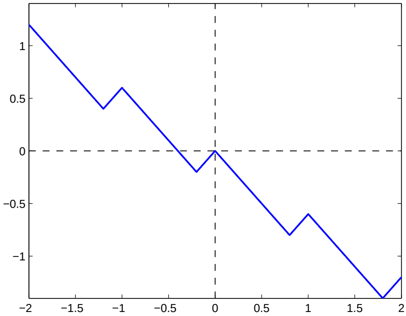

Figure: $\phi_{0.8}(a)$

n-step conic MIR inequalities

$$Q^n := \left\{(y, w^+, w^-, t) \in \mathbb{Z} \times \mathbb{Z}_+^{n-1} \times \mathbb{R}_+^3 : \left|\sum_{i=1}^n \alpha_i y_i + w^+ - w^- - \beta\right| \leq t\right\}.$$

- *n*-step conic MIR inequality: valid inequality for Q^n
- Streamlined validity proof
- Facet-defining under certain conditions
- *k*-step conic MIR inequality for Q^n $(1 \leq k < n)$

$$S = \left\{x \in \mathbb{Z}_+^N, t \in \mathbb{R} : \left|\sum_{j \in J} a_j x_j + w^+ - w^- - b\right| \leq t\right\}$$

- *n*-step conic MIR inequality for S: $\sum_{j \in J} \phi^{\alpha,b}(a_j) y_j - \phi^{\alpha,b}(b) \leq t$
- $\phi^{\alpha,b}(u)$: *n*-step conic MIR function
- $\phi^{\alpha,b}(u) = u - 2\mu^n_{\alpha,b}(u)$
- $\mu^n_{\alpha,b} : \mathbb{R} \to \mathbb{R}$ is the *n*-step MIR function (Kianfar and Fathi, 2009)

Examples of ϕ

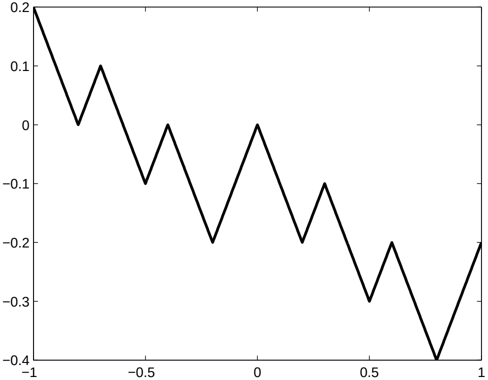

Figure: $\phi^{\alpha,0.8}(u), \alpha = (1, 0.3)$

Examples of ϕ

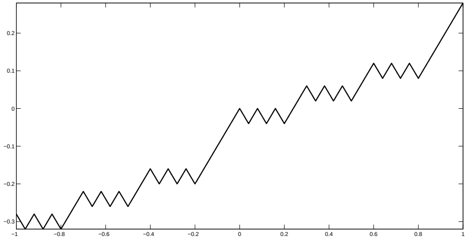

Figure: $\phi^{\alpha,0.8}(u), \alpha = (1, 0.3, 0.08)$

Examples of ϕ

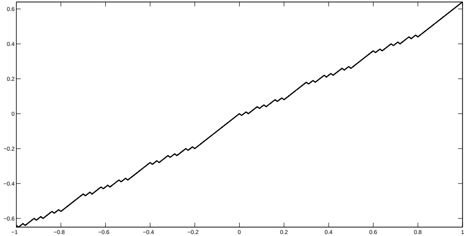

Figure: $\phi^{\alpha,0.8}(u), \alpha = (1, 0.3, 0.08, 0.03)$

A new perspective

Consider the sets

$$K_1 = \{(y,z) \in \mathbb{Z}^n \times \mathbb{R} : (a_2 - a_1)y + 2z \geq (b_2 - b_1), z \geq 0\}$$
$$K_2 = \{(x,s) \in \mathbb{Z}^n \times \mathbb{R} : a_1 x + s \geq b_1, a_2 x + s \geq b_2\}$$

Main Theorem

The inequality

$$\pi y + z \geq \pi_0$$

is valid (facet-defining) for K_1 if and only if the inequality

$$(2\pi + a_1)x + s \geq (2\pi_0 + b_1)$$

if valid (facet-defining) for K_2.

A new perspective

Proof Idea.

- $K_2 = \{(x, s) \in \mathbb{Z}^n \times \mathbb{R} : a_1 x + s \geq b_1, a_2 x + s \geq b_2\}$
- New variable, $z = 0.5(a_1 x + s - b_1)$
- $s = 2z + b_1 - a_1 x$
- First constraint becomes $2z \geq 0$
- Second constraint becomes $(a_2 - a_1)x + 2z \geq b_2 - b_1$
- $K_1 = \{(x, z) \in \mathbb{Z}^n \times \mathbb{R} : (a_2 - a_1)x + 2z \geq (b_2 - b_1), z \geq 0\}$
- Affine independence preserved
- VI for K_2: $\pi x + z \geq \pi_0$
- Substitute $z = 0.5(a_1 x + s - b_1)$
- VI for K_1: $(2\pi + a_1)x + s \geq 2\pi_0 + b_1$

An Example

$$K_1 = \{(y, z) \in \mathbb{Z}^n \times \mathbb{R} : y + 2z \geq 0.4, z \geq 0\}$$
$$K_2 = \{(x, s) \in \mathbb{Z}^n \times \mathbb{R} : x + s \geq 0.8, 2x + s \geq 1.2\}$$

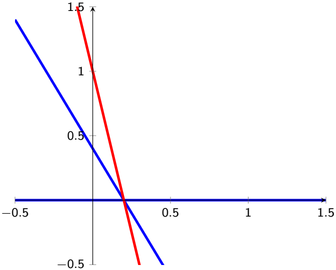

Figure: K_1

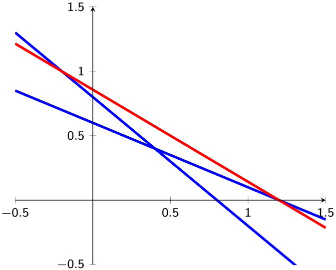

Figure: K_2

Knapsack and PSOC sets

$$K_1 = \{(y,z) \in \mathbb{Z}^n \times \mathbb{R} : (a_2 - a_1)y + 2z \geq (b_2 - b_1), z \geq 0\}$$
$$K_2 = \{(x,s) \in \mathbb{Z}^n \times \mathbb{R} : a_1 x + s \geq b_1, a_2 x + s \geq b_2\}$$

Set $a_1 = -a,\ a_2 = a,\ b_1 = -b,\ b_2 = b$.

$$K_1 = \{(y,z) \in \mathbb{Z}^n \times \mathbb{R} : 2ay + 2z \geq 2b, z \geq 0\}$$
$$K_2 = \{(x,s) \in \mathbb{Z}^n \times \mathbb{R} : -ax + s \geq -b, ax + s \geq b\}$$

or

$$K_1 = \{(y,z) \in \mathbb{Z}^n \times \mathbb{R} : ay + z \geq b, z \geq 0\}$$
$$K_2 = \{(x,s) \in \mathbb{Z}^n \times \mathbb{R} : |ax - b| \leq s\}$$

Knapsack and PSOC sets

$$K_1 = \{(y, z) \in \mathbb{Z}^n \times \mathbb{R} : ay + z \geq b, z \geq 0\}$$
$$K_2 = \{(x, s) \in \mathbb{Z}^n \times \mathbb{R} : |ax - b| \leq s\}$$

Theorem

The inequality

$$\pi y + z \geq \pi_0$$

is valid (facet-defining) for K_1 if and only if the inequality

$$(2\pi - a)x + s \geq 2\pi_0 - b$$

is valid (facet-defining) for K_2.

Simple conic MIR (again)

$$K_1 = \{(y,z) \in \mathbb{Z} \times \mathbb{R} : y + z \geq \beta, z \geq 0\}$$
$$K_2 = \{(x,s) \in \mathbb{Z} \times \mathbb{R} : |x - \beta| \leq s\}$$

Simple conic MIR inequality for K_2: $(1-2f)(x - \lfloor \beta \rfloor) + f \leq s \quad (f = \beta - \lfloor \beta \rfloor)$.

1-step MIR inequality for K_1:

$$\begin{aligned} fy + z &\geq f\lceil \beta \rceil \qquad (f = \beta - \lfloor \beta \rfloor) \\ \pi y + z &\geq \pi_0 \end{aligned}$$

Corresponding inequality for K_2:

$$\begin{aligned} (2f-1)x + s &\geq 2f\lceil \beta \rceil - \beta \\ (2\pi - a)x + s &\geq 2\pi_0 - b \end{aligned}$$

Rearranging, $\quad (1-2f)(x - \lfloor \beta \rfloor) + f \leq s$

An Example

$$K_1 = \{(y, z) \in \mathbb{Z} \times \mathbb{R} : y + z \geq 0.8, z \geq 0\}$$
$$K_2 = \{(x, s) \in \mathbb{Z} \times \mathbb{R} : |x - 0.8| \leq s\}$$

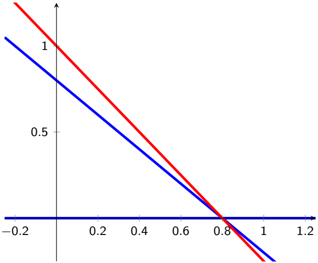

Figure: 1-step MIR set

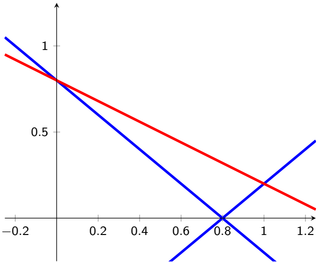

Figure: 1-step conic MIR set

n-step conic MIR (again)

$$K = \{y \in \mathbb{Z}_+^N, z \in \mathbb{R} : \sum_{j=1}^N a_j y_j + z \geq b, z \geq 0\}$$

$$S = \{x \in \mathbb{Z}_+^N, s \in \mathbb{R} : \left|\sum_{j=1}^N a_j x_j - b\right| \leq s\}$$

n-step conic MIR inequality for S: $\sum_{j=1}^N (a_j - 2\mu(a_j))x_j - (b - 2\mu(b)) \leq s$

n-step MIR inequality for K (Kianfar and Fathi, 2009):

$$\begin{aligned} \sum_{j=1}^N \mu(a_j)y_j + z &\geq \mu(b) \\ \pi y + z &\geq \pi_0 \end{aligned}$$

Corresponding inequality for S:

$$\begin{aligned} \sum_{j=1}^N (2\mu(a_j) - a_j)x_j + s &\geq 2\mu(b) - b \\ (2\pi - a)x + s &\geq 2\pi_0 - b \end{aligned}$$

Rearranging, $\sum_{j=1}^N (a_j - 2\mu(a_j))x_j - (b - 2\mu(b)) \leq s$

Mingling Inequalities

$$\overline{K}_1 = \{(y, z) \in \mathbb{Z}^n_+ \times \mathbb{R}_+ : ay + z \geq b, y \leq u\}$$

- Mingling inequalities for $\overline{K}_1$ (Atamtürk and Günlük , 2010)
- *n*-step mingling inequalities for $\overline{K}_1$ (Atamtürk and Kianfar, 2012)

$$\overline{K}_2 = \{(x, s) \in \mathbb{Z}^n_+ \times \mathbb{R} : |ax - b| \leq s, x \leq u\}$$

Theorem

The *n-step conic mingling inequality* is valid for $\overline{K}_2$ and facet-defining under certain conditions.

Knapsack and PSOC sets (contd...)

$$K_2 = \{(x, s) \in \mathbb{Z}^n \times \mathbb{R} : |ax - b| \leq s\}$$
$$K_3 = \{(y, w^+, w^-, v) \in \mathbb{Z}^n \times \mathbb{R}^2_+ \times \mathbb{R} : |ay + w^+ - w^- - b| \leq v\}.$$

Theorem

The inequality

$$\pi x + \pi_0 \leq s$$

is valid (facet-defining) for K_2 if and only if the inequality

$$\pi y + \pi_0 \leq v + w^+ + w^-$$

is valid (facet-defining) for K_3.

Polyhedral reformulation revisited

Set with conic constraint:

$$X = \left\{(x,z) \in \mathbb{Z}^n_+ \times \mathbb{R}^p_+ : \|Ax + Gz - b\| \leq dx + ez - h\right\}$$

Reformulation (Atamtürk and Narayanan, 2010):

$$\begin{aligned} t_0 &\leq dx + ez - h \\ t_i &\geq |a_i x + g_i z - b| \quad i = 1, \ldots, m \\ t_0 &\geq \|t\| \end{aligned}$$

Simpler form of PSOC constraint: $t_i \geq |a_i x + w_i^+ - w_i^- - b|,\ w_i^+, w_i^- \geq 0$

Multi-row Sets

$$K_{1m} = \{(y, z) \in \mathbb{Z}_+^n \times \mathbb{R}_+^m : a_i y + z_i \geq b_i, i = 1, \ldots, m\}$$
$$K_{2m} = \{(x, s) \in \mathbb{Z}_+^n \times \mathbb{R}^m : |a_i x - b_i| \leq s_i, i = 1, \ldots, m\}$$

Theorem

The inequality

$$\pi y + \sum_{i=1}^{m} c_i z_i \geq \pi_0$$

is valid (facet-defining) for K_{1m} if and only if the inequality

$$\left(2\pi - \sum_{i=1}^{m} c_i a_i\right) x + \sum_{i=1}^{m} c_i s_i \geq 2\pi_0 - \sum_{i=1}^{m} c_i b_i$$

is valid (facet-defining) for K_{2m}.

Multi-row Sets

$$K_{2m} = \{(x, s) \in \mathbb{Z}_+^n \times \mathbb{R}^m : |a_i x - b_i| \leq s_i, i = 1, \ldots, m\}$$
$$K_{3m} = \{(y, w^+, w^-, z) \in \mathbb{Z}_+^n \times \mathbb{R}_+^{3m} : |a_i y + w_i^+ - w_i^- - b_i| \leq z_i, i = 1, \ldots, m\}.$$

Theorem

The inequality

$$\pi x + \pi_0 \leq \sum_{i=1}^{m} c_i s_i$$

is valid (facet-defining) for K_{2m} if and only if the inequality

$$\pi y + \pi_0 \leq \sum_{i=1}^{m} c_i \left(z_i + w_i^+ + w_i^-\right)$$

is valid (facet-defining) for K_{3m}.

Main References

1. Atamtürk , A., Narayanan, V.: Conic mixed-integer rounding cuts. *Mathematical Programming Ser. A* **122**, 1–20, 2010.
2. Sanjeevi, S., Masihabadi, S., Kianfar, K.: *n*-step conic mixed integer rounding inequalities. *Mathematical Programming* (submitted), (2012).