Mixing n-step MIR Inequalities

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Outline

1-Mixing set [Günlük and Pochet, 2001]

$$Q^{m,1} = \{(y^1, \ldots, y^m, v) \in \mathbb{Z}^m \times \mathbb{R}_+ : \alpha_1 y^i + v \geq \beta_i, i = 1, \ldots, m\}$$

- Multi-constraint set
- Each constraint has 1 integer variable
- substructure in lot-sizing, facility location, network design MIPs

MIR inequalities

Constraint i:

$$\alpha_1 y^i + v \geq \beta_i$$

Define $\beta^{(1)} := \beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor$

MIR inequality associated with constraint i:

$$v \geq \beta_i^{(1)} \left(\left\lceil \frac{\beta_i}{\alpha_1} \right\rceil - y^i \right)$$

Mixing inequalities

- For $K \subseteq \{1, ..., m\}$, WLOG let $K = \{1, ..., k\}$ such that ${\beta_{i-1}}^{(1)} \leq {\beta_i}^{(1)}, i = 2, \ldots, k$.
- MIR inequalities for constraints in K "mixed" to get mixing inequalities.

$$v \geq \sum_{i=1}^{k} \left(\beta_i^{(1)} - \beta_{i-1}^{(1)}\right)\left(\left\lceil \frac{\beta_i}{\alpha_1} \right\rceil - y^i\right)$$

$$v \geq \sum_{i=1}^{k} \left(\beta_i^{(1)} - \beta_{i-1}^{(1)}\right)\left(\left\lceil \frac{\beta_i}{\alpha_1} \right\rceil - y^i\right) + \left(\alpha_1 - \beta_k^{(1)}\right)\left(\left\lceil \frac{\beta_1}{\alpha_1} \right\rceil - y^1 - 1\right).$$

Mixing inequalities

- Mixing inequalities describe convex hull of $Q^{m,1}$.
- Valid inequalities for:
  - Single capacity lot-sizing
  - Single capacity facility location
  - Capacitated network design
  - Multiple knapsack
  - Simplex tableau

Variants of $Q^{m,1}$

- Two divisible coefficients
- Two non-divisible coefficients
- n divisible coefficients
- Mixing set with flows
- Mixing sets linked by bidirected paths

n-step MIR inequalities [Kianfar and Fathi, 2009]

n-step MIR inequalities [Kianfar and Fathi, 2009]

Developed for general set $\{(x, s) \in \mathbb{Z}_+^{|J|} \times \mathbb{R}_+ : \sum_{j \in J} a_j x_j + s \geq b\}$

Special case:

$$Q^{1,n} = \{(y_1, \ldots, y_n, v) \in \mathbb{Z} \times \mathbb{Z}_+^{n-1} \times \mathbb{R}_+ : \sum_{j=1}^{n} \alpha_j y_j + v \geq \beta\}.$$

Define recursive remainder $\beta^{(j)} = \beta^{(j-1)} - \alpha_j \left\lfloor \beta^{(j-1)} / \alpha_j \right\rfloor$, where $\beta^{(0)} := \beta$

Assume $\alpha_j \left\lceil \beta^{(j-1)} / \alpha_j \right\rceil \leq \alpha_{j-1}$, $j = 2, \ldots, n$.

n-step MIR inequalities

n-step MIR inequality for $Q^{1,n}$:

$$v \geq \beta^{(n)} \left(\prod_{l=1}^{n} \left\lceil \frac{\beta^{(l-1)}}{\alpha_l} \right\rceil - \sum_{j=1}^{n} \prod_{l=j+1}^{n} \left\lceil \frac{\beta^{(l-1)}}{\alpha_l} \right\rceil y_j \right).$$

Define integer-valued linear function $\phi : \mathbb{Z}^n \to \mathbb{Z}$

$$\phi(y) := \prod_{l=1}^{n} \left\lceil \frac{\beta_i{}^{(l-1)}}{\alpha_l} \right\rceil - \sum_{j=1}^{n} \prod_{l=j+1}^{n} \left\lceil \frac{\beta_i{}^{(l-1)}}{\alpha_l} \right\rceil y_j \quad \text{for } i \in K$$

Compact form:

$$v \geq \beta^{(n)} \phi(y)$$

n-mixing set

Generalized mixing set

$$Q^{m,n} = \{(y, v) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+ : \sum_{j=1}^{n} \alpha_j y_j^i + v \geq \beta_i, i = 1, \ldots, m\}$$

Assumed condition: $\alpha_j \left\lceil \beta_i^{(j-1)} / \alpha_j \right\rceil \leq \alpha_{j-1}$, $j = 2, \ldots, n$.

n-step MIR inequality associated with constraint i:

$$v \geq \beta_i^{(n)} \phi^i(y^i)$$

where $y^i = (y_1^i, y_2^i, ..., y_n^i)$.

Mixed n-step MIR inequalities

For $K \subseteq \{1, ..., m\}$, let $K = \{1, ..., k\}$ such that

$\beta_{i-1}{}^{(n)} \leq \beta_i{}^{(n)}, i = 2, \ldots, k.$

Mixed n-step MIR inequalities generated by K:

$$v \geq \sum_{i=1}^{k} \left(\beta_i{}^{(n)} - \beta_{i-1}{}^{(n)}\right) \phi^i(y^i),$$

$$v \geq \sum_{i=1}^{k} \left(\beta_i{}^{(n)} - \beta_{i-1}{}^{(n)}\right) \phi^i(y^i) + \left(\alpha_n - \beta_k{}^{(n)}\right) \left(\phi^1(y^1) - 1\right).$$

Properties

- Valid for $Q^{m,n}$.
- Facet-defining for $Q^{m,n}$.
- Validity conditions always hold when $\alpha_n|\alpha_{n-1}|...|\alpha_2|\alpha_1$.
- Multi-row valid inequalities for general MIPs.
- New valid inequalities for special structure MIPs.

Valid inequalities for general MIPs

$$Y_m = \{(x_1, \dots, x_N, s) \in \mathbb{Z}_+^N \times \mathbb{R}_+^m : \sum_{j \in J} a_{ij} x_j + s_i \geq b_i, i = 1, \dots, m\}$$

For $K \subseteq \{1, ..., m\}$, let $K = \{1, ..., k\}$ such that $b_{i-1}{}^{(n)} \leq b_i{}^{(n)}, i = 2, \dots, k$.

For parameters $(\alpha_1, \alpha_2, ..., \alpha_n)$ such that $\alpha_j \left\lceil b_i{}^{(j-1)} / \alpha_j \right\rceil \leq \alpha_{j-1}$, $j = 2, \dots, n$, define integer-valued linear function $\sigma^n_{\alpha,b} : \mathbb{R}^k \to \mathbb{R}$.

Valid inequalities for general MIPs

Let $a_j = (a_{1j}, a_{2j}, ..., a_{kj})$, $b = (b_1, b_2, ..., b_k)$.

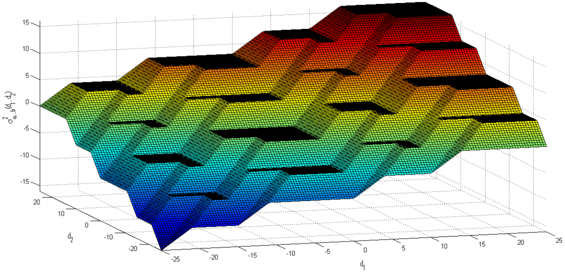
Mixed n-step MIR inequality for Y_m:

$$\sum_{j \in J} \sigma^n_{\alpha,b}(a_j)x_j + \bar{s} \geq \sigma^n_{\alpha,b}(b)$$

where $\bar{s} \geq s_i$ for $i \in K$.

σ function

$\sigma^2_{(25,10),(39,18)}(d_1, d_2)$ for $(d_1, d_2) \in [-25, 25] \times [-16, 16]$:



Lot-sizing with Multi-Capacity Modules (LMM)

- $T := \{1, ..., m\}$ time periods
- $(\alpha_1, \alpha_2, ..., \alpha_n)$ n available capacity module sizes
- x_t - production, s_t - inventory, z_t^j - number of modules of α_j
- d_t - demand in period t

$$X^{LMM} = \Big\{ (x, s, z) \in \mathbb{R}_+^m \times \mathbb{R}_+^m \times \mathbb{Z}_+^{m \times n} :$$

$$s_{t-1} + x_t = d_t + s_t, \qquad t \in T$$

$$x_t \leq \sum_{j=1}^{n} \alpha_j z_t^j, \qquad t \in T \quad \Big\}$$

Valid inequalities for LMM

Mixing inequalities:

$$\bar{v} \geq \sum_{i=1}^{|I|} \left({b_i}^{(n)} - {b_{i-1}}^{(n)} \right) \phi^i(y^i),$$
$$\bar{v} \geq \sum_{i=1}^{|I|} \left({b_i}^{(n)} - {b_{i-1}}^{(n)} \right) \phi^i(y^i) + \left(\alpha_n - {b_{|I|}}^{(n)} \right) \left(\phi_n^1(y^1) - 1 \right)$$

- $\bar{v}$ - linear function of inventory and production variables
- y^i - linear functions of z_j^i variables
- b - set of demands in a subset of T

generalize capacity constraints of the form $x_t \leq C_t z_t$ and divisible capacity modules

Valid inequalities for LMM

- generalize (k, l, S, I) inequalities (Pochet and Wolsey, 1993) to multi-capacity case
- special case 1: capacity constraints of the form $x_t \leq C_t z_t$
- special case 2: $C_n | C_{n-1} | ... | C_1$

Multi-capacity Facility Location (MFL)

P - set of facilities, Q - set of clients, $(\alpha_1, \alpha_2, ..., \alpha_n)$ - capacity modules for facilities

x_{pq} - demand of client q satisfied by facility p

u_p^j - number of capacity modules installed in facility p

$$X^{MFL} = \Big\{ (x,u) \in \mathbb{R}_+^{n_P n_Q} \times \{0,1\}^{n_P n} : \\ \sum_{p \in P} x_{pq} = d_q, \qquad q \in Q \\ \sum_{q \in Q} x_{pq} \leq \sum_{j=1}^{n} \alpha_j u_p^j, \qquad p \in P \quad \Big\}.$$

Valid inequalities for MFL

$$\sum_{(p,q)\in T} x_{pq} \geq \sum_{i=1}^{n_l} \left(b_i{}^{(n)} - b_{i-1}{}^{(n)}\right) \phi^i(y^i),$$

$$\sum_{(p,q)\in T} x_{pq} \geq \sum_{i=1}^{n_l} \left(b_i{}^{(n)} - b_{i-1}{}^{(n)}\right) \phi^i(y^i) + \left(\alpha_n - b_{n_l}{}^{(n)}\right) \left(\phi_n^1(y^1) - 1\right)$$

b_i, y^i defined based on parameters and decision variables of MFL

generalize valid inequalities of Aardal, Pochet and Wolsey, 1995

Future research

- Properties of mixing inequalities for general and special structure MIPs
- Other special structures
- Computational experiments

Thank you.