n-step Conic Mixed Integer Rounding Cuts

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SOCMIP

Second-order Conic Mixed Integer Program:

$$\min\ cx + ry$$
$$\text{s.t. } \|A_i x + G_i z - b_i\| \leq d_i x + e_i z - h_i, \quad i = 1, \ldots, k,$$
$$x \in \mathbb{Z}^n, z \in \mathbb{R}^p$$

- A_i, G_i, and b_i have m_i rows
- d_i, e_i, c and r are vectors and h_i is a scalar
- Rational data

Features

- Second Order Cone Program (SOCP) with some integer variables
- generalizes LP, QCQP
- Applications in portfolio optimization, signal processing
- Conic constraints - polyhedral reformulation (Atamtürk and Narayanan, 2010)
- Valid inequalities for reformulation $\Rightarrow$ VIs for SOCMIP

PSOC

Set with conic constraint:

$$X = \left\{(x, z) \in \mathbb{Z}^n_+ \times \mathbb{R}^p_+ : \|Ax + Gz - b\| \leq dx + ez - h\right\}$$

Reformulation:

$$t_0 \leq dx + ez - h \tag{1}$$

$$t_i \geq |a_i x + g_i z - b| \quad i = 1, \ldots, m \tag{2}$$

$$t_0 \geq \|t\| \tag{3}$$

Inequality (2): Polyhedral second-order conic (PSOC) constraint

Simple conic MIR inequality

$$S_0 := \left\{ (x, \omega^+, \omega^-, t) \in \mathbb{Z} \times \mathbb{R}^3_+ : |x + \omega^+ - \omega^- - b| \leq t \right\}.$$

Simple conic MIR inequality (Atamtürk and Narayanan, 2010)

$$(1 - 2f)\,(x - \lfloor \beta \rfloor) + f \leq t + \omega^+ + \omega^-$$

- valid for S_0, cuts off all points in $relax(S_0)$
- generates nonlinear inequalities for S_0, valid inequalities for general PSOC sets

Our research contribution

$$Q^n := \{(y, \omega^+, \omega^-, t) \in \mathbb{Z} \times \mathbb{Z}_+^{n-1} \times \mathbb{R}_+^3 : \left| \sum_{i=1}^{n} \alpha_i y_i + \omega^+ - \omega^- - \beta \right| \leq t\}.$$

- n-step conic MIR inequality: facet-defining for Q^n
- facets for Q^n from n_1-step conic MIR inequalities $(1 \leq n_1 < n)$
- nonlinear inequalities for Q^n
- new valid inequalities for more general PSOC sets
- n-step MIR inequalities are conic n-step MIR inequalities

n-step conic MIR inequality

Let $\beta^{(j)} := \beta^{(j-1)} - \alpha_n \left\lfloor \beta^{(j-1)}/\alpha_n \right\rfloor$, $\beta^{(0)} = \beta$.

Assume

- $\beta^{(j-1)}/\alpha_j \notin \mathbb{Z}$
- $\alpha_j \left\lceil \beta^{(j-1)}/\alpha_j \right\rceil \leq \alpha_{j-1}$

for $j = 2, \ldots, n$.

n-step conic MIR inequality for Q^n

$$\sum_{i=1}^{n} \left(\alpha_i - 2\beta^{(n)} \prod_{k=i+1}^{n} \left\lceil \frac{\beta^{(k-1)}}{\alpha_k} \right\rceil \right) \left(y_i - \left\lfloor \frac{\beta^{(i-1)}}{\alpha_i} \right\rfloor \right) + \beta^{(n)} \leq t + \omega^+ + \omega^-. \quad (CMIR_n)$$

facet-defining for Q^n.

More facets for Q^n

VI based on n_1-step conic MIR inequality ($1 \leq n_1 < n$)

$$\sum_{i=1}^{n_1} \left(\alpha_i - 2\beta^{(n_1)} \prod_{k=i+1}^{n_1} \left\lceil \frac{\beta^{(k-1)}}{\alpha_k} \right\rceil \right) \left(y_i - \left\lfloor \frac{\beta^{(i-1)}}{\alpha_i} \right\rfloor \right) + \beta^{(n_1)}$$

$$\leq t + \sum_{i=n_1+1}^{n} \alpha_i y_i + \omega^+ + \omega^-$$

Facet-defining if $\left\lfloor \beta^{(i-1)}/\alpha_i \right\rfloor \geq 1$ for $i = n_1 + 1, \ldots, n$

Nonlinear inequalities

Theorem

Let the conditions $\alpha_j \left\lceil \beta^{(j-1)}/\alpha_j \right\rceil \leq \alpha_{j-1}$ hold for $j = 2, ..., n$. The inequality

$$\left| \sum_{i=1}^{n} \left(\alpha_i - 2\beta^{(n)} \prod_{k=i+1}^{n} \left\lceil \frac{\beta^{(k-1)}}{\alpha_k} \right\rceil \right) \left(y_i - \left\lfloor \frac{\beta^{(i-1)}}{\alpha_i} \right\rfloor \right) + \beta^{(n)} \right|$$

$$\leq t + \omega^{+} + \omega^{-}$$

is valid for Q^n if and only if $\left\lceil \beta^{(i-1)}/\alpha_i \right\rceil = \alpha_{i-1}/\alpha_i$ for $i = 2, \ldots, n$.

General PSOC sets

$$S := \left\{ x \in \mathbb{Z}_+^N, z^+, z^- \in \mathbb{R}_+, t \in \mathbb{R} : \left| \sum_{j \in J} a_j x_j + z^+ - z^- - b \right| \leq t \right\}$$

- For $n \in \mathbb{N}$, choose $\overline{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{R}^n$.
- $\alpha_i > 0$ for $i = 1, ..., n$ and $b^{(i-1)}/\alpha_i \notin \mathbb{Z}$ for $i = 2, ..., n$.
- Define $\phi^{\overline{\alpha}, b} : \mathbb{R} \to \mathbb{R}$

General PSOC sets

Theorem

Given a parameter vector $\overline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$, where $\alpha_i > 0$ for all i, the inequality

$$\sum_{j \in J} \phi^{\overline{\alpha},b}(a_j) y_j - \phi^{\overline{\alpha},b}(b) \leq t + z^+ + z^-$$

is valid for S if $\alpha_i \left\lceil b^{(i-1)} / \alpha_i \right\rceil \leq \alpha_{i-1}$ for $i = 2, \ldots, n$.

Properties of ϕ:

- continuous, superadditive
- $\phi^{\overline{\alpha},b}(u) = u - 2\mu^n_{\overline{\alpha},b}(u)$

Examples of ϕ

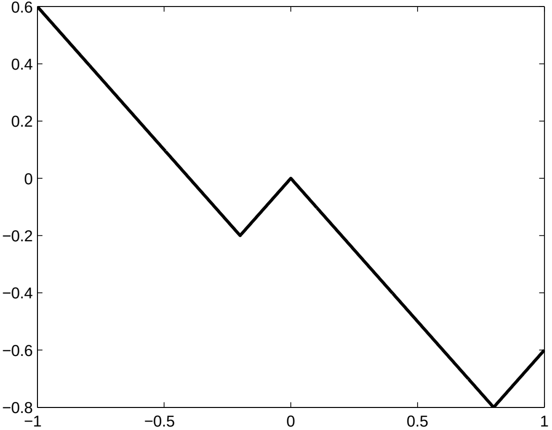

Figure: $\phi^{\overline{\alpha},0.8}(u), \overline{\alpha} = 1$

Examples of ϕ

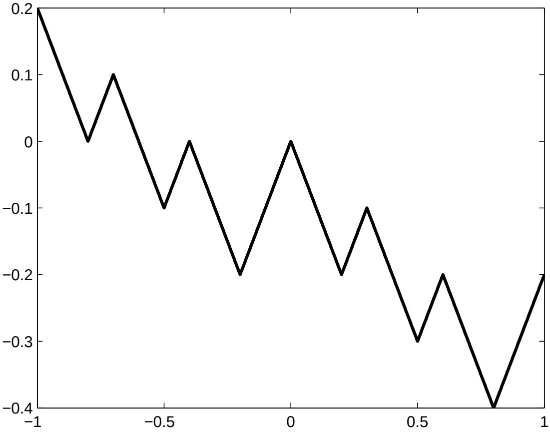

Figure: $\phi^{\overline{\alpha},0.8}(u), \overline{\alpha} = (1, 0.3)$

Examples of ϕ

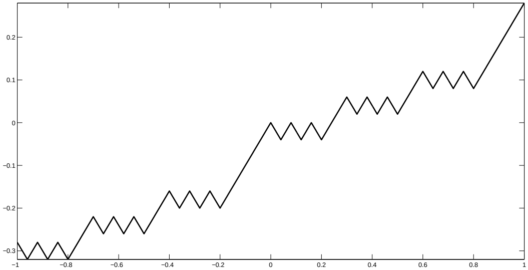

Figure: $\phi^{\overline{\alpha},0.8}(u), \overline{\alpha} = (1, 0.3, 0.08)$

Examples of ϕ

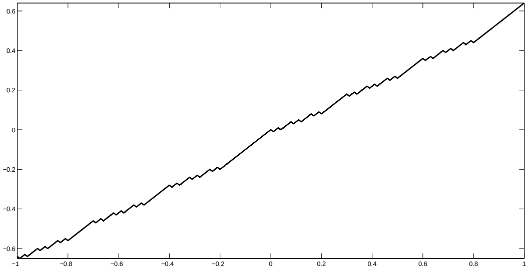

Figure: $\phi^{\overline{\alpha},0.8}(u), \overline{\alpha} = (1, 0.3, 0.08, 0.03)$

Relation to n-step MIR

Linear inequalities as PSOC constraints (Atamtürk and Narayanan, 2010)

Linear inequalities:

$$\alpha_1 x + \beta_1 y \leq \gamma_1 \text{ and } \alpha_2 x + \beta_2 y \leq \gamma_2$$

PSOC constraint:

$$\left| \left(\frac{\alpha_1 - \alpha_2}{2}\right) x + \left(\frac{\beta_1 - \beta_2}{2}\right) y - \left(\frac{\gamma_1 - \gamma_2}{2}\right) \right| \leq \left(\frac{\gamma_1 + \gamma_2}{2}\right) - \left(\frac{\alpha_1 + \alpha_2}{2}\right) x + \left(\frac{\beta_1 + \beta_2}{2}\right) y.$$

Relation to n-step MIR

- n-step MIR inequalities :
- facet-defining for $\{(x,s) \in \mathbb{Z}_+^{|J|} \times \mathbb{R}_+ : \sum_{j \in J} a_j x_j + s \geq b\}$
- $\sum_{j \in J} a_j x_j + s \geq b$ and $s \geq 0$
- write as PSOC constraint
- n-step conic MIR inequality for this constraint $\rightarrow$ n-step MIR inequality

Main References

1. Atamtürk , A., Narayanan, V.: Conic mixed-integer rounding cuts. *Mathematical Programming Ser. A* **122**, 1–20 (2010).