A Polyhedral Study of Triplet Formulation for Single Row Facility Layout Problem

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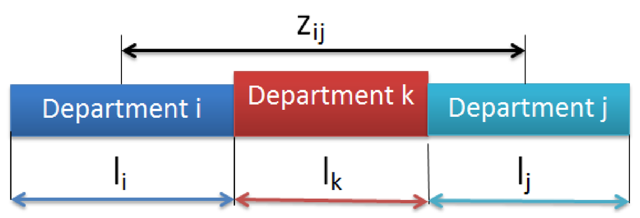
Single Row Facility Layout Problem

Single Row Facility Layout Problem (SRFLP): linear arrangement problem with the objective of minimizing the total weighted sum of distances between department pairs.



- l_i - Length of department i
- c_{ij} - Average daily traffic between departments i and j
- z^{π}_{ij} - Distance between centroids of departments i and j in permutation π

$$\textit{Objective of SRFLP:}\ \min_{\pi} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c_{ij} z^{\pi}_{ij}$$

- Amaral's formulation of SRFLP - triplet formulation [1].
- Two projections of triplet formulation and its valid inequalities presented in [1].
- Linear program solved over valid inequalities in [1] yields optimal solution for several classical instances of sizes $n = 5$ to $n = 30$.
- This suggests that the valid inequalities are quite strong.

Research Contributions

- Dimension of the triplet polytope is $n'' = n(n-1)(n-2)/3$.
- Almost all valid inequalities defined for the triplet polytope by Amaral in [1] are facet-defining.
- The above results are also true for the other projections of the triplet polytope defined in [1].

Triplet Polytope

- Decision variable:

$$\lambda_{ijk} = \begin{cases} 1 & \text{if department } k \text{ lies between departments } i \text{ and } j,\ i<j \\ 0 & \text{otherwise.} \end{cases}$$

- Set of departments: $N = \{1, ..., n\}$
- Decision variable vector: $\lambda = \{\lambda_{ijk} : i,j,k \in N, i<j\}$
- Number of elements of λ: $n' = n(n-1)(n-2)/2$
- $P^1 = \{\lambda \in \{0,1\}^{n'} : \lambda$ represents a permutation of $\{1, ..., n\}\}$
- *Triplet polytope:* convex hull of P^1.

$$\textit{Objective function of SRFLP:}\ \min \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c_{ij} \left(\frac{1}{2}(l_i + l_j) + \sum_{k \neq i, k \neq j}^{n} l_k \lambda_{ijk} \right)$$

Valid inequalities for SRFLP presented in [1]:

$$0 \leqslant \lambda_{ijk} \leqslant 1 \qquad i,j,k \in N, i<j \qquad (1)$$
$$\lambda_{ijk} + \lambda_{ikj} + \lambda_{jki} = 1 \qquad i,j,k \in N, i<j<k \qquad (2)$$
$$-\lambda_{ijd} + \lambda_{jkd} + \lambda_{ikd} \geqslant 0 \qquad i,j,k,d \in N, i<j<k \qquad (3)$$
$$\lambda_{ijd} + \lambda_{jkd} - \lambda_{ikd} \geqslant 0 \qquad i,j,k,d \in N, i<j<k \qquad (4)$$
$$\lambda_{ijd} - \lambda_{jkd} + \lambda_{ikd} \geqslant 0 \qquad i,j,k,d \in N, i<j<k \qquad (5)$$
$$\lambda_{ijd} + \lambda_{jkd} + \lambda_{ikd} \leqslant 2 \qquad i,j,k,d \in N, i<j<k \qquad (6)$$

For a positive even integer $\beta \leqslant n$, let $S = \{i_t : t = 1, \ldots, \beta\} \subseteq N$ be a set of departments and $d \in S$. Let (S_1, S_2) be a partition of $S \backslash \{d\}$ such that $|S_1| = \beta/2$. Then the following inequality is valid for $conv(P^1)$ [1]:

$$\sum_{p,q \in S_1 : p<q} \lambda_{pqd} + \sum_{p,q \in S_2 : p<q} \lambda_{pqd} \leqslant \sum_{p \in S_h, q \in S_{\{1,2\} \backslash h} : h=1,2, p<q} \lambda_{pqd} \qquad (7)$$

Preliminary Results

Notation:

- Π_N - set of all permutations of departments in N.
- λ^{π} - vector that represents a permutation $\pi \in \Pi_N$.
- To simplify notation, wherever we have λ_{ijk} with $i > j$, we mean λ_{jik}.

Lemma 1. *For given distinct departments $x, y, z \in N$, let $\pi^1, \pi^2, \pi^3, \pi^4$ be four permutations of the departments in N satisfying the following conditions:*

- π^1: *First two departments are x,y*
- π^2: *First two departments are y,x, other departments are in the same order as π^1*
- π^3: *First three departments are z,x,y*
- π^4: *First three departments are z,y,x, other departments are in the same order as π^3*

If the λ vectors corresponding to these permutations lie on the hyperplane

$$\sum_{i,j,k \in N : i<j} a_{ijk} \lambda_{ijk} = b, \qquad (8)$$

then $a_{yzx} = a_{xzy}$.

Outline of proof:

Substitute λ^{π^1} in (8) | Substitute λ^{π^2} in (8) | Substitute λ^{π^3} in (8) | Substitute λ^{π^4} in (8)

Compare LHS and cancel terms | Compare LHS and cancel terms

$$\sum_{h \neq x,y} \boldsymbol{a_{xhy}} = \sum_{h \neq x,y} \boldsymbol{a_{yhx}}$$

$$\boldsymbol{a_{yzx}} + \sum_{h \neq x,y,z} \boldsymbol{a_{xhy}} = \boldsymbol{a_{xzy}} + \sum_{h \neq x,y,z} \boldsymbol{a_{yhx}}$$

$$\boldsymbol{a_{yzx} = a_{xzy}}$$

Dimension of $conv(P^1)$

Theorem 2. *$conv(P^1)$ is of dimension $n'' = n(n-1)(n-2)/3$.*

$conv(P^1) \subset \mathbb{R}^{n'}$ and any $\lambda \in P^1$ satisfies the set of $\binom{n}{3}$ linearly independent equalities (2). Hence, dim $\left(conv(P^1)\right) \leqslant n' - \binom{n}{3} = n''$. To prove that the dimension is actually equal to n'', we just need to show that any other hyperplane like

$$\sum_{i,j,k \in N : i<j} a_{ijk} \lambda_{ijk} = b \qquad (9)$$

satisfied by all $\lambda \in P^1$ will be a linear combination of the equalities (2).

Outline of proof:

$\lambda^{\pi} \in P^1$ for any permutation π, by definition of P^1.

For distinct departments x, y, z, choose any $\pi^1, \pi^2, \pi^3, \pi^4$ satisfying the conditions of Lemma 1.

Since $\pi^1, \pi^2, \pi^3, \pi^4$ lie on (9), by Lemma 1,
$\boldsymbol{a_{xzy} = a_{yzx}}$

Using these permutations, construct 4 other permutations by replacing departments x by y, y by z and z by x.

Since these 4 permutations lie on (9), by Lemma 1,
$\boldsymbol{a_{xyz} = a_{xzy}}$

$\boldsymbol{a_{xyz} = a_{xzy} = a_{yzx}}$

Substituting in (9),
$$\boldsymbol{b} = \sum_{i,j,k \in N : i<j} \boldsymbol{a_{ijk}}$$

Hence, (9) becomes
$$\sum_{i,j,k \in N : i<j} \boldsymbol{a_{ijk}[\lambda_{ijk} + \lambda_{ikj} + \lambda_{jki}]} = \sum_{i,j,k \in N : i<j} \boldsymbol{a_{ijk}}$$
Thus, it is a linear combination of equalities (2).

Facet-defining property of valid inequalities

Lemma 3. *Consider inequality (7) for given β, S, S_1, S_2 and d. Let $\pi \in \Pi_N$, and γ_1 and γ_2 be the number of departments in S_1 and S_2 which are to the left of d in π, respectively. Then $\lambda^{\pi} \in P^1$ satisfies (7) at equality if and only if $\gamma_1 - \gamma_2 = 0$ or 1.*

Let $|S_1| = \alpha$.
Hence, $|S_2| = \alpha - 1$.

Number of departments in S_1 to the left of $d = \gamma_1$, to the right of $d = \alpha - \gamma_1$.

Number of departments in S_2 to the left of $d = \gamma_2$, to the right of $d = \alpha - 1 - \gamma_2$.

First summation in LHS of (7): $\gamma_1(\alpha - \gamma_1)$
Second summation in LHS of (7): $\gamma_2(\alpha - 1 - \gamma_2)$
Summation of RHS of (7): $\gamma_1(\alpha - 1 - \gamma_2) + \gamma_2(\alpha - \gamma_1)$

The validity of (7) is equivalent to,
$\gamma_1(\alpha - \gamma_1) + \gamma_2(\alpha - 1 - \gamma_2) \leq \gamma_1(\alpha - 1 - \gamma_2) + \gamma_2(\alpha - \gamma_1)$

This reduces to,
$(\gamma_1 - \gamma_2) \leq (\gamma_1 - \gamma_2)^2$.
This shows that for (7) to be satisfied at equality, $\gamma_1 - \gamma_2 = 0$ *or* 1.

Theorem 4. *Any of inequalities (7) is facet-defining for $conv(P^1)$.*

Let P' be the face of $conv(P^1)$ defined by an inequality of the form (7) for given β, S, S_1, S_2 and d. Hence, for every point in P', (7) is satisfied at equality, i.e.

$$\sum_{p,q \in S_1 : p<q} \lambda_{pqd} + \sum_{p,q \in S_2 : p<q} \lambda_{pqd} - \sum_{p \in S_1, q \in S_2} \lambda_{pqd} = 0. \qquad (10)$$

To prove that (10) is a facet, we need to show that any hyperplane like

$$\sum_{e,f,g \in N : e<f} a_{efg} \lambda_{efg} = b \qquad (11)$$

that passes through P' is a linear combination of hyperplanes (2) and hyperplane (10).

Using lemma 1 and lemma 3, we can prove that
$a_{efg} = a_{egf} = a_{fge}$ for any e, f, g such that $d \notin \{e,f,g\}$ *or* $\{e,f,g\} \not\subset S$
$a_{pdq} = a_{qdp}$ *for all* $p, q \in S_1$
$a_{sdt} = a_{tds}$ *for all* $s, t \in S_2$
$a_{pds} = a_{sdp}$ *for all* $p \in S_1, s \in S_2$

Equation (11) reduces to a linear combination of (2) and
$$\sum_{p,q \in S_1, p<q} [a_{pqd} - a_{pdq}] \lambda_{pqd} + \sum_{s,t \in S_2, s<t} [a_{std} - a_{sdt}] \lambda_{std} + \sum_{p \in S_1, s \in S_2} [a_{psd} - a_{pds}] \lambda_{psd} = b_1$$

By considering other permutations satisfying the conditions of lemma 3 and substituting their λ vectors in (11) and the previous equation, we can prove that $b_1 = 0$ and
$a_{pqd} - a_{pdq} = a_{std} - a_{sdt} = -(a_{psd} - a_{pds}) = K$ *for all* $p, q \in S_1$ *and* $s, t \in S_2$

Equation (11) reduces to a linear combination of (2) and
$$K \left(\sum_{p,q \in S_1, p<q} \lambda_{pqd} + \sum_{s,t \in S_2, s<t} \lambda_{std} - \sum_{p \in S_1, s \in S_2} \lambda_{psd} \right) = 0$$
This proves that (11) is a linear combination of equalities (2) and (10).

- (3), (4) and (5) - special cases of (7) for $\beta = 4$, so they are facet-defining.
- (1) and (6) not facet-defining in general - can be checked numerically.
- We have also shown that the above results are true for other projections of the triplet polytope in [1] using a simple result related to projection of polyhedra [2].

Concluding Remarks

Our results provide theoretical support for the fact that the LP solution over these valid inequalities gives the optimal solution for all instances studied in [1].

Major References

1. Amaral, A. R. S. 2009. A new lower bound for the single row facility layout problem. *Discrete Applied Mathematics* **157(1)**, 183–190.
2. Sanjeevi, S. and Kianfar, K. 2010. A polyhedral study of triplet formulation for single row facility layout problem. *Discrete Applied Mathematics*, **in press**, DOI: 10.1016/j.dam.2010.07.005.